A NEW PARAMETERS AND DIFFERENTIAL EQUATIONS OF ROTATION IN ORIENTATION OF A RIGID BODY PROBLEM

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Kinematics and dynamic differential equations of rotation of a rigid body for the «non-classical» conjugate three- dimensional new vectors of rotation are considered in this article. The vectors modules contain tangent and cotangent of one-fourth of the rotation angle. This paper also delivers applications of the conjugate equations in the dynamic and of orientation problems of a rigid body. Different new (polar) dynamical differential equations of RB rotation come out on the basis of kinematics equations.

In classical *Euler* occasion, for example, problem of decision of the system of six dynamical differential equations of *Euler-Poisson* reduces to integration of the system of just three dynamical equations with two independent classical first integrals of energy and square. Three new "tangent" vector coordinates-parameters definitely determine RB orientation in contrast to three direction cosines determined orientation of just one unit vector of supporting basis *I* in *Poisson* equations.

New vectors and conjugated polar equations define a (essentially) new line of fundamental research in solid mechanics.

Keywords: rotation vector Rodrigues, equations Euler-Poisson, orientation, rigid body, group rotation.

1. Three-dimensional vectors of rotation with minimal possible quantity of generalized coordinate lining with three freedoms of rigid body (RB)[1-4] with fixed point are of the most interest in the problem of dimensional orientation such as a task of orientation order parameter identification and directing of rotation (spherical) motion of RB.

Classical ***Rodrigues*** vector [1,4] has a module with tangent ($tg(\varphi/2)$) of half-angle φ of finite rotation of RB. This vector can not be used in the condition of $\varphi=\pi(180°)$ in the problems of RB dimensional orientation. Considered in the works [2-4] tangent, cotangent *conjugate rotation vectors* $\boldsymbol{\tau}=\tau\boldsymbol{k}$, $\boldsymbol{\rho}=\rho\boldsymbol{k}$, where $\tau=k_\tau tg(\varphi/4)$, $\rho=k_\rho ctg(\varphi/4)$ (k_τ, k_ρ — arbitrary constant coefficients), $\boldsymbol{k}$ — unit vector of ***Euler*** axis of RB finite rotation are of the most interest (by $0<\varphi<2\pi$) because of its conjugate features [2,4,5].

2. The article deals with *conjugative polar* non-linear vector kinematics differential equations of RB rotation in the transformation of lineal vector $\boldsymbol{\omega}(t)$ of RB angular velocity [3,4]

$$\boldsymbol{\tau}^* = \tau\,'\boldsymbol{\Theta}_\tau\,\boldsymbol{\omega},\;\; \boldsymbol{\rho}^* = \rho\,'\boldsymbol{\Theta}^T_\rho\,\boldsymbol{\omega}, \qquad (1)$$

where $\boldsymbol{\tau}^* = d\boldsymbol{\tau}/dt$, $\boldsymbol{\rho}^* = d\boldsymbol{\rho}/dt$ - local derivative of vectors in the time t (derivative towards some connected with RB coordinate base); $\tau\,' = \partial\tau/\partial\varphi$, $\rho' = \partial\rho/\partial\varphi$ - particular derived modules τ, ρ of vectors of angle φ, defined as function

$\tau' = (k_\tau + \tau^2/k_\tau)/4$, $\rho' = -(k_\rho + \rho^2/k_\rho)/4$; $\boldsymbol{\Theta}_\tau$, $\boldsymbol{\Theta}^T_\rho$ - orthogonal operators *semi-rotation* of RB (angle $\varphi/2$) [3]:

$$\boldsymbol{\Theta}_\tau = \boldsymbol{E} + 2(k_\tau \boldsymbol{T} + \boldsymbol{T}^2)/(k^2_\tau + \tau^2),$$

$$\boldsymbol{\Theta}^{\mathrm{T}}_\rho = \boldsymbol{E} + 2(-k_\rho \boldsymbol{R} + \boldsymbol{R}^2)/(k_\rho^2 + \rho^2),$$

«T» - transposing.

Therein $\boldsymbol{T}$, $\boldsymbol{R}$ - skew-symmetric operators of vector multiplication [2, 4], satisfies the identities $\boldsymbol{T\tau} = \boldsymbol{\tau} \times \boldsymbol{\tau} = \boldsymbol{0}$, $\boldsymbol{R\rho} = \boldsymbol{\rho} \times \boldsymbol{\rho} = \boldsymbol{0}$ - vector products, $\boldsymbol{0}$ - zero vector, $\boldsymbol{E}$ - unit operator. Operators $\boldsymbol{\Theta}_\tau$, $\boldsymbol{\Theta}_\rho$ satisfy the identity $(\boldsymbol{\Theta}_\tau\boldsymbol{\Theta}_\rho)^2 = \boldsymbol{E}$, determined conjugacy of vectors $\boldsymbol{\tau}$, $\boldsymbol{\rho}$ and equations (1) [4] as features of “duality” and isomorphic correspondence.

3. Equations (1) have the first («trigonometric») integral in the form of scalar product $(\boldsymbol{\tau} \cdot \boldsymbol{\rho}) = \tau\,\rho = k_\tau k_\rho = C$ (arbitrary constant) and admit simple and graphic kinematics interpretation - polar precession-nutation model [4] with arbitrary vector $\boldsymbol{\omega}_\tau$ and for any RB. In the first equation vector $\boldsymbol{\omega}$ is transformed (at every moment of the traveling time of RB) by operator $\boldsymbol{\Theta}_\tau$ into vector $\boldsymbol{\omega}_\tau$ as a result of precession $\boldsymbol{\omega}_\tau$ on the angle $\psi_\tau = \varphi/2$ (rotation vector $\boldsymbol{\omega}$ with module $\omega(t)$ about the surface of some circular cone of «velocity» precession cone angle $2\upsilon_\omega$) around ***Euler*** axis with unit vector $\boldsymbol{k}$. Angle υ_ω - nutation angle (vector deviation $\boldsymbol{\omega}$ from the unit vector $\boldsymbol{k}$). And then a vector $\boldsymbol{\omega}_\tau$ is multiplied by scalar operator $\tau'\boldsymbol{E}$. At the second equation vector $\boldsymbol{\omega}$ precesses (rotates n the surface of that) at the angle $\psi_{\Theta\rho} = (\pi + \varphi/2)$ and is multiplied by scalar operator $\rho'\boldsymbol{E}$. Angle υ_ω of nutation is determined from scalar product $(\boldsymbol{\omega} \cdot \boldsymbol{\tau})$, $(\boldsymbol{\tau}^* \cdot \boldsymbol{\tau})$ or $(\boldsymbol{\omega} \cdot \boldsymbol{\rho})$, $(\boldsymbol{\rho}^* \cdot \boldsymbol{\rho})$.

On the basis of such interpretation equation - models (1) come out, for example, kinematics anholonomic equality:

$$(\boldsymbol{a} \cdot \boldsymbol{b})^2 + ((\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{a} \times \boldsymbol{b})) = (\boldsymbol{a} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{b}),$$

where $\boldsymbol{a} = \boldsymbol{\omega} \times \boldsymbol{\tau}$, $\boldsymbol{b} = (\boldsymbol{\Theta}_\tau\,\boldsymbol{\omega}) \times \boldsymbol{\tau}$.

Equations (1) have general decisions in the form of ***Cauchy*** [4, 5]. General decision of the first equation in (1) determines the form shape of the new vector representations of three-dimensional rotation (a ***Lie*** group) [6]. In this group inverse element $\boldsymbol{\tau}^{-1}$ is equal to adverse vector, i. e. $\boldsymbol{\tau}^{-1} = -\boldsymbol{\tau}$, but unit element is equal to zero vector $\boldsymbol{0}$.

4. Different new (polar) dynamical differential equations of RB rotation come out on the basis of kinematics equations (1).

In classical ***Euler*** occasion, for example, problem of decision of the system of six dynamical differential equations of ***Euler-Poisson*** [7] reduces to integration of the system of just three dynamical equations with two independent classical first integrals (of energy and square [7]). These two integrals are enough to consider the system of three equations to be integrated [7].

At this case change is introduced in the equations (1) $\boldsymbol{\omega} = \boldsymbol{S}^{-1}\boldsymbol{g}$, where $\boldsymbol{g}$ - constant vector of kinetic moment (constant module and in the line of supporting basis I [7]); $\boldsymbol{S}^{-1}$ - inverse operator ***S*** of RB inertia [3,4,7] (constant in connected with RB basis ***J***).

Contained of (1), for example, polar matrix differential equation with vector coordinates-parameters $\boldsymbol{\tau}$ has the form (see also [3]):

$$\tau^*_{J} = \tau\,'(\Theta_{\tau J}\ S_J^{-1}\ \Theta_{\tau J}^{\ \mathrm{T}})\ \Theta_{\tau J}^{\ \mathrm{T}} g_{\ I}\,, \qquad (2)$$

where $\tau^*_{J} = [\ \dot{\tau_j}_1\ \dot{\tau_j}_2\ \dot{\tau_j}_3\ ]^{\mathrm{T}}$ - column matrix with derivative coordinates $\boldsymbol{\tau}^*$ in basis ***J***; $\Theta_{\tau J}$ - matrix (3x3) of operator $\boldsymbol{\Theta}_\tau$ in basis ***J***; S_J - diagonal matrix (3x3) with three constant the main moments of RB inertia; S_J^{-1} - inverse matrix, $g_I = [g_{i1}\ g_{i2}\ g_{i3}]^{\mathrm{T}}$ - constant column matrix with vector coordinates ***g*** in basis ***I***.

Three coordinates of the new vector-parameter $\boldsymbol{\tau}$ is uniquely determined RB orientation (as opposed to the three direction cosines of only one unit ortho-vertical supporting basis ***I*** in the ***Poisson*** equations [7]). In the problems of the dynamics of RB and synthesis problems of attitude control laws (definition of control moments) with the use of ***Lyapunov*** functions (see, eg, [8]), the scalar equations (1), (2) are used in conjunction with the classical dynamic ***Euler*** equations [1, 7, 8].

5. Conjugate vectors $\boldsymbol{\tau}$, $\boldsymbol{\rho}$ are considering as parts of vectors with non-traditional non-normalized (non-***Hamiltonian***) rotation new quaternions [8]. Such non-normalized quaternions can be effectively used instead of classical normalized quaternions (with parameters of ***Rodrigues-Hamilton***, ***Euler-Rodrigues*** [4]) in different «particular» tasks of inertial orientation [1, 4, 9, 10], including RB orientation control problem [8].

Sets of coordinates-parameters of conjugate vectors $\boldsymbol{\tau}$, $\boldsymbol{\rho}$, corresponded to them quaternions sufficiently broaden the circle of new RB orientation parameters. Such parameters can be of interest in the problem of global parameterization [9] of three-dimensional rotation group and new representation ***Lorenz*** group in quantum mechanics [11].

New vectors $\boldsymbol{\tau}$, $\boldsymbol{\rho}$ and conjugated polar equations (1), (2) define the (essentially) the new direction of fundamental research in solid mechanics.

References

1. Branetz V.N., Shmyglevsky I.P. Introduction into Theory of Strapdown Intentional Navigation System. [in Russian], Nauka. Moskow. Main Redaction of Physical- and Mathematics lit. 1992. - 280 p.
2. Panov A.P. Kinematics Differential Equations for the Eigenvectors of Rotation Operators of a Rigid Body // Izvestija AN SSSR. MTT [Mechanic of Solids], - 1985. - Vol.20. № 4. - p. 26-32. - THE ALLERTOL PRESS JOURNAL PROGRAMM, Inc. New York. 1985.
3. Panov A.P. Polar Forms of Vector Equations of Rigid Body Rotation. // Ibidem. - 1991. №5. p. 19-25. - THE ALLERTOL PRESS JOURNAL PROGRAMM, Inc. New York. 1991.

4. Panov A.P. Mathematical Foundations of Inertial Orientation Theory. [in Ukraine], Naukova Dumka, - Kyiv, 1995. - 279 p.
5. Panov A.P. On the construction of general solution of kinematic vector rotation equetions // Izvestija AN SSSR. MTT [Mechanic of Solids], №4. - p. 51-56. 1987. - THE ALLERTOL PRESS JOURNAL PROGRAMM, Inc. New York. 1987.
6. Panov A.P. Rotation Groups and their Lie Algebras in Problems of Rigid Body Orientation // Cybernetics and Computing. - 1993. - Issue. 99. - p. 7 - 15. – THE ALLERTOL PRESS JOURNAL PROGRAMM, Inc. New York. 1993.
7. Arhangelsky U.A. Analytic Dynamic of Rigid Body. [in Russian], Nauka, Moscow. 1977. - 328 p.
8. Panov A.P. On the New Non-Normalized Rotation Quaternions of Solids // Problem of Analytical Mechanics and its Application / Proceedings of the Institute of Mathematics of the NAS of Ukraine. - Vol. 26. Kiyv. - 1999. - p. 300-329.
9. Panov A.P. On the Applying of Non-Normalized Quaternions of Five-Dimension Rotation Vectors and their Algebras in Inertial Orientation Problem // VIII International Scientific and Technical Conference Gyro Technologies, Navigation, Motion Control and Design of Aircraft and Spacecraft. Book of Proceedings/ Part 1. NTUU “Kyiv Polytechnic Institute”, 21-22 of April 2011, p. 131-137. http://faks.kpi.ua/library/Conferences/Conference-8/Section1/paper_21.pdf .
10. Panov A.P., Tsysarzh V.V., Aksenov V.V. On the new quaternion-based Methods of Solving Guidance, Navigation and Control Problems for Strapdown Inertial Reference Systems//VII Saint-Petersburg International Conference on Integrated Navigation Systems. Abstract. Saint-Petersburg. 2000. p. 115-117.
11. Fedorov, Lorenz Group, [in Russian], Nauka, Moskow, 1979, - 384p.