NONTANGENTIAL CONVERGENCE OF THE GENERALIZED POISSON-ABEL MEANS

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Abstract. The means of Fourier series $U(f,y;\lambda,h)$ generated by semi-continuous summation methods $\Lambda = \{\lambda_k(h),\ k = 0,1,...;\ h > 0\}$ are studied. For the points (y,h), belonging to an angular domain $\Gamma_d(x)$, upper L^p-estimates of the corresponding maximal operators are obtained. Nontangential convergence almost everywhere of the generalized Poisson-Abel means, corresponding to a case of $\lambda_k(h) = \exp(-hk^\alpha),\ \ k = 0,1,...;\ \ \alpha \ge 1$, is established.

Keywords. Exponential summation methods, estimates of L^p -norms, nontangential convergence.

1. Introduction. Formulation of the problem. Let $L_{2\pi}$ be class of 2π-periodical functions, which are summable on $[-\pi, \pi]$ and $C^2(0, +\infty)$ – class of functions having continuous second derivative on $(0,+\infty)$. In this paper we consider the semi-continuous means

$$U(f,y;\lambda,h) = \sum_{k=-\infty}^{\infty} \lambda_{|k|}(h) c_k(f) \exp(iky) \tag{1}$$

of Fourier series $s[f]$ of functions $f \in L_{2\pi}$. In the definition (1)

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \exp(-ikt)\, dt, \quad k = 0, \pm 1, \pm 2, ...$$

are complex Fourier coefficients of function f.

We study the problem of behavior (1) at $(y,h) \to (x,0)$, when the point (y,h) is within the boundaries of the angular domain

$$\Gamma_d(x) = \{(y,h) \mid y \in [-\pi, \pi],\ h > 0,\ \frac{|y-x|}{h} \le d\},\ d = const,\ d > 0.$$

The case of "radial" convergence $U(f,x;\lambda,h) \to f(x)$ at $h \to +0$ was investigated in [1].

2. The main result. Define

$$U_*(f) = U_*(f,x;\lambda) = \sup_{(y,h)\in\Gamma_d(x)} |U(f,y;\lambda,h)|;$$

let $m = [\frac{1}{2dh}]$ and

$$f^*(x) = \sup_{\eta>0} \frac{1}{2\eta} \int_{x-\eta}^{x+\eta} |f(t)|\, dt$$

be Hardy maximal function ([2], vol.1, p.55).

Theorem 1. Let the sequence $\{\lambda_N(h)\}$ decreases so rapidly that

$$N|\lambda_N(h)| + N^2 |\Delta\lambda_N(h)| = o(1),\ N \to \infty, \tag{2}$$

and there is a constant $C = C_{\Lambda,d}$ such that

$$\sum_{k=0}^{\infty} \frac{(m+k+1)(k+1)}{m} \,|\, \Delta^2 \lambda_k(h) \,| \le C. \tag{3}$$

Then for every x the estimate

$$U_*(f,x;\lambda) \le C_{\Lambda,d} \;\; f^*(x)$$

holds.

Here and throughout the paper C will represent constants, which depend only on the explicitly specified indexes.

3. L^p -estimates. Let

$$\| f \|_p = (\int_{-\pi}^{\pi} | f(x) |^p dx)^{1/p}$$

be a norm in Lebesgue space L^p ($p > 0$; $L = L^1$; $\| f \| = \| f \|_1$).

Theorem 2. If the sequence Λ satisfies the conditions (2) and (3), the following estimates

$$\| U_*(f) \|_p \le C_{p,\Lambda} \, \| (f) \|_p, \;\; p > 1;$$

$$\| U_*(f) \| \le C_\Lambda (1 + \| f(\ln^+ | f |) \|) \;\;;$$

$$\| U_*(f) \|_p \le C_{p,\Lambda} \, \| (f) \|, \;\; 0 < p < 1. \tag{4}$$

hold.

3. Nontangential convergence.

Theorem 3. If $f \in L_{2\pi}$, the sequence Λ satisfies (2), (3) and

$$\lim_{h \to 0} \lambda_k(h) = 1, \;\; k = 0,1,\ldots,$$

then the relation

$$\lim_{\substack{(y,h) \to (x,0) \\ (y,h) \in \Gamma_d(x)}} U(f,y;\lambda,h) = f(x)$$

holds almost everywhere.

This theorem can be proved by the standard method ([2], vol. 2, pp. 464-465) due to the estimate (4).

4. Exponential means. Denote now

$$\lambda_0(h) = 1, \qquad \lambda_k(h) = \lambda(x,h)|_{x=k}, \;\; k = 1,2,\ldots,$$

where $\lambda(x,h) = \exp(-h\varphi(x))$, and require the following conditions:

A) $\varphi \in C^2(0,+\infty)$; $\varphi(x) \ge 0$, $\varphi'(x) \ge 0$, $\varphi''(x) \ge 0$, $x \in (0,+\infty)$;

B) $x^2 (\varphi'(x))^2 \exp(-h\varphi(x))$ and $x^2 | \varphi''(x) | \exp(-h\varphi(x))$ decrease to zero as x increases.

Note that

$$\lambda_x''(x,h) = h \exp(-h\varphi(x))(h(\varphi'(x))^2 - \varphi''(x)) \qquad .$$

and apply twice the Lagrange theorem to the second finite differences in (3).

Under the conditions of B) the sum of (3) is majorized by a corresponding improper integral and for implementability of statements of Theorems 1, 2, 3 it is sufficient to require

$$\int_0^{\infty} (h^2(\varphi'(x))^2 + h \,|\, \varphi''(x) \,|)(x + hx^2) \;\exp(-h\varphi(x)) \;\, dx \le C_\varphi .$$

5. Generalized Poisson-Abel means. Consider in particular the case of $\varphi(x) = x^\alpha$, $\alpha \ge 1$, then

$$\lambda_0(h) = 1, \;\; \lambda_k(h) = \exp(-hk^\alpha), \;\; k = 1,2,\ldots; \;\; \alpha \ge 1.$$

Corollary 1. The statements of Theorems 2 and 3 are valid for generalized Poisson-Abel means

$$\sigma(f,y;\alpha,h)=\sum_{k=-\infty}^{\infty}\exp(-h\,|k\,|^{\alpha})c_k(f)\exp(iky)$$

for all $\alpha \geq 1$; the constants C in the estimates of L^p-norms is $C = C_{\alpha,p}$.

In particular, the relation

$$\lim_{\substack{(y,h)\to(x,0)\\(y,h)\in\Gamma_d(x)}}\sum_{k=-\infty}^{\infty}\exp(-h\,|k\,|)c_k(f)\exp(iky)=f(x)\,,\quad f\in L_{2\pi}\quad,$$

(nontangential convergence of Poisson-Abel means) holds for almost all x.

6. Exponentially-polynomial summation methods. Let now $\varphi(x)$ is a polynomial function of n-th degree

$$P_n(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_0,\quad a=a_n>0\,;\quad n=1,2,\ldots$$

Corollary 2. The assertions of Theorems 2 and 3 are valid for exponentially-polynomial means

$$\sigma(f,y;n,h)=\sum_{k=-\infty}^{\infty}\exp(-hP_n(|k\,|))c_k(f)\exp(iky)$$

for all $n=1,2,\ldots$; the constants C in the estimates of L^p-norms is $C = C_{n,p}$.

References

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